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Non-Birman–Wenzl algebraic properties and redundancy of exotic enhanced Yang–Baxter operator for spin model

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Abstract. In this paper we explicitly construct the Markov trace for the general coloured exotic braid group representations (BGR) with spin-j. It is verified that the BGR for $j=1$ is redundant in the sense of Murakami, but no Birman–Wenzl algebra.

1. Introduction

Some of the trigonometric solutions of the Yang–Baxter equation (YBE) can be derived through the Yang–Baxterization prescription [1, 2] for given braid group representations, which are related to the quantum algebra through either the quantum double of Drinfeld [3] or matrix Hopf algebra [4]. The Yang–Baxterization approach [4, 5] is based on the number of distinct eigenvalues of a considered BGR denoted by S. The physical solutions of YBE should satisfy the boundary condition

$$\check{R}(x=0)=\text{constant}\times S \tag{1.1}$$

initial condition

$$\check{R}(x=1)=\text{constant}\times I \tag{1.2}$$

and the unitarity condition

$$\check{R}(x)\check{R}(x^{-1})=\rho(x)I \tag{1.3}$$

where $\rho(x)$ is a scalar function of spectral parameter x, and I is the unity matrix. In [4, 5] we found that such a Yang–Baxterization prescription is related with the untangling properties of a BGR. An interesting example is the Birman–Wenzl algebra (BWA). In [5, 6] we pointed out that if a BGR obeys BWA then the Yang–Baxterization prescription works sufficiently [1, 5, 6]. It is equivalent to the Baxterization of Jones [1] for the standard case. It is also verified that our non-standard solutions associated with $B(n)$, $C(n)$ and $D(n)$ all belong to BWA [6].

On the other hand, one meets the Alexander link polynomials [8, 13] for some non-standard solutions of BGR. In this case invariant tangles appear instead of the usual links.

Following Murakami [12] if an enchanced Yang–Baxter operator (YBE) Y is redundant then invariant tangles associated with the corresponding BGR exist.

We have proved that any BGR belonging to BWA must be redundant [6]. Furthermore, in [6, 7] it has been shown that our non-standard solutions of BGR associated with $B(n)$, $C(n)$ and $D(n)$ obey BWA, and some BGR associated with $C(n)$ and $D(n)$ lost the definition of loop in Kauffman state model. Therefore they are redundant in the sense of Murakami [12] and there exist invariant tangles [6]. A question is then naturally raised: whether there is a BGR with three distinct eigenvalues, which is not BWA but still redundant.

The answer is yes. In this paper we shall explore the following points.

(1) The exotic solution for spin 1 presented in [13] is definitely not BWA.

(2) A general Markov trace theory is set up for our non-standard solutions with spin j derived by the representations of quantum algebra with q a root of unity [14, 16]. The solutions given in [12, 13] are special cases of our solutions.

(3) By direct calculation we prove that the exotic BGR for spin 1 is redundant so that the associated invariant tangles are rigorously defined.

2. Yang–Baxterization and properties of BGR with three distinct eigenvalues

We first review the general scheme of Yang–Baxterization for a BGR denoted by S with three distinct eigenvalues

$$(S-\lambda_1)(S-\lambda_2)(S-\gamma_3)=0. \tag{2.1}$$

It is proved in [2, 5] that if S satisfies

$$f=f_3^+\theta_3^+ + f_3^-\theta_3^- + f_2\theta_2 + f_1^+\theta_1^+ + f_1^-\theta_1^- = 0 \tag{2.2}$$

where

$$\begin{aligned}\theta_3^\pm &= S_1^{\pm1}S_2^{\mp}S_1^{\mp1} - S_2^{\pm1}S_1^{\mp1}S_2^{\pm1}\\ \theta_2 &= S_1S_2^{-1} - S_2S_2^{-1} + S_2^{-1}S_1 - S_2^{-1}S_2\\ \theta_1^\pm &= S_1^{\pm1} - S_2^{\pm1}\end{aligned} \tag{2.3}$$

and $f_3^\pm, f_2, f_1^\pm$ are given by

$$\begin{aligned} &f_3^+ = \frac{\lambda_1}{\lambda_3^2}, \qquad f_3^- = -\frac{\lambda_1^2}{\lambda_3}\\ &f_2 = -\frac{\lambda_1}{\lambda_3}\left(1+\frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda_3}+\frac{\lambda_1}{\lambda_3}\right) \qquad f_1^\pm = \mp\lambda_2^{\mp1}f_2\end{aligned} \tag{2.4}$$

then $\check{R}(x)$ satisfies YBE

$$\check{R}_1(x)\check{R}_2(xy)\check{R}_1(y) = \check{R}_2(y)\check{R}_1(xy)\check{R}_2(x) \tag{2.5}$$

if $\check{R}(x)$ is constructed by

$$\check{R}(x) = A(x)S + B(x)I + C(x)S^{-1} \tag{2.6}$$

with

$$A(x)-\lambda_3^{-1}(x-1) \qquad C(x)=\lambda_1 x(x-1)$$
$$B(x)=\left(1+\frac{\lambda_1}{\lambda_2}+\frac{\lambda_1}{\lambda_3}+\frac{\lambda_2}{\lambda_3}\right)x. \tag{2.7}$$

As has been shown in [5], if S obeys the BWA then S satisfies

$$f(\lambda_1, \lambda_2, \lambda_3)=0 \tag{2.8}$$

and

$$f(\lambda_2, \lambda_1, \lambda_3)=0. \tag{2.9}$$

One can not derive that S obeys BWA even if S satisfies both (2.8) and (2.9)

Without loss of generality the eigenvalues of S are taken as $\lambda_1=\lambda$, $\lambda_2=\lambda^{-1}$ and $\lambda_3=l^{-1}$ and $m=\lambda+\gamma^{-1}$ then we have:

Proposition 1. If S satisfies (2.8) then (2.9) is equivalent to

$$E_1S_2E_1-E_2S_1E_2=l(E_1-E_2) \tag{2.10}$$

where

$$S_i+S_i^{-1}=m(I+E_i)(i=1, 2). \tag{2.11}$$

Proof. By the definition of E_i, equation(2.1) can be recast into

$$E_i^2=\{m^{-1}(l+l^{-1})-I\}E_i \tag{2.12}$$

or

$$S_i^2=m(S_i+l^{-1}E_i)-I. \tag{2.13}$$

Substituting (2.11)-(2.13) into (2.2) we obtain

$$\begin{aligned} f(\lambda_1, \lambda_2, l)=f(\lambda, \lambda^{-1}, l) &= -l^2m^2(E_1-E_2)-\lambda^2lm^2(E_1S_2E_1-E_2S_1E_2)+\lambda l^2m(S_1E_2S_1-S_2E_1S_2) \\ &\quad +\lambda l^2m^2(E_1S_2+S_2E_1-E_2S_1-S_1E_2). \end{aligned} \tag{2.14}$$

Using (2.14), a direct check gives

$$\lambda^{-1}f(\lambda, \lambda^{-1}, l)-\lambda f(\lambda^{-1}, \lambda, l)=(\lambda-\lambda^{-1})lm^2\{E_1S_2E_1-E_2S_1E_2-l(E_1-E_2)\}. \tag{2.15}$$

Thus $f(\lambda_1, \lambda_2, \lambda_3)=0$ then $f(\lambda_2, \lambda_1, \lambda_3)$ vanishes if and only if (2.10) holds.

Since (2.10) cannot determine $E_1S_2E_1=lE_1$ which is the point for the existence of BWA, we conclude that both (2.8) and (2.9) are not enough to determine BWA. However if (2.8) holds but (2.10) does not, this immediately determines that S must not be BWA.

Proposition 2. The following BGR is not BWA:

$$\bar{S}=\text{block diag }(A_1, A_2, A_3, A_4, A_5) \tag{2.16}$$

where

$$A_1=1 \qquad A_5=\omega t^4$$

$$A_2=\begin{bmatrix}0 & t\\ t & 1-t^2\end{bmatrix} \qquad A_4=\begin{bmatrix}0 & \omega^2t^3\\ \omega^2t^3 & t^2(1-\omega t^2)\end{bmatrix}$$

$$A_3=\begin{bmatrix}0 & 0 & t^2\\ 0 & \omega t^2 & \mathrm{i}\omega tZ\\ t^2 & \mathrm{i}\omega tZ & (1-t^2)(1-\omega t^2)\end{bmatrix}$$

with $\omega^3=1$ and $Z=((1-t^2)(1-\omega t^2))^{\frac{1}{2}}$.

The distinct eigenvalues are

$$\lambda_1=1, \qquad \lambda_2=-t^2, \qquad \lambda_3=\omega t^4 \tag{2.17}$$

To prove proposition 2 we substitute (2.16) and (2.17) into $f(\lambda_2,\lambda_1,\lambda_3)$. The calculation shows that $f(\lambda_2,\lambda_1,\lambda_3)\neq 0$. For instance the element

$$\{f(\lambda_2,\lambda_1,\lambda_3)\}_{012}^{102}=(\omega t^8+\omega^2t^4+1)t^{-9} \tag{2.18}$$

vanishes only when $t=1$

Thus by proposition 1 the $\bar{S}$ given by (2.16) does not obey BWA. Therefore, $\bar{S}$ can be Yang–Baxterized by (2.6) though $\bar{S}$ does not obey BWA This example is interesting because the BGR given by (2.16) is not BWA, but the Yang–Baxterization prescription still works. This solution differs from the super-extended BGR associated with $B(n)$, $C(n)$ and $(D(n)$ that obey BWA. As was shown in [14] the super-case corresponds to genetic q-representation of quantum algebra, whereas (2.16) comes from quantum algebra at q, a root of unity, and its $U(1)$ extension is permitted by the quantum double.

3. Markov properties of exotic BGR for spin model

In general it is difficult to perform the parameter extension in preserving the quantum double since the explicit representation should be used. However, for $SL_q(2)$ one can use the q-deformed Holstein–Primakoff transformation to make such an extension. Obviously the extension means more parameters appearing in the associated BGR, hence we shall obtain new solutions of the BGR.

In our previous work [14, 16] the q-boson realizations of $SL_q(2)$ with more parameters were established. For $SL_q(2)$ q-algebra

$$[\hat{J}_+,\hat{J}_-]=[\hat{J}_0] \tag{3.1}$$

$$[\hat{J}_0,\hat{J}_\pm]=\pm 2\hat{J}_\pm \tag{3.2}$$

where $[n]=(q^n-q^{-n})/(q-q^{-1})$, a mapping can be defined through

$$\hat{J}_+\to J_+=a^+\alpha(\hat{N}) \tag{3.3}$$

$$\hat{J}_-\to J_-=a^-\beta(\hat{N}) \tag{3.4}$$

$$\hat{J}_0\to J_0=2\hat{N}-\lambda \tag{3.5}$$

where

$$\alpha(\hat{N}-1)\beta(\hat{N})=[\lambda+1-\hat{N}] \tag{3.6}$$

$$[n]=(q^n-q^{-n})/q-q^{-1}) \tag{3.7}$$

and

$$a^+a^-=[\hat{N}] \qquad a^-a^+=[\hat{N}+1] \qquad [\hat{N}, a^\pm]=a^\pm. \tag{3.8}$$

On the Fock space we then have

$$\begin{aligned} &J_+|n\rangle=\alpha(n)|n+1\rangle \qquad J_0|n\rangle=(2n-\lambda)|n\rangle \\ &J_-|n\rangle=[n]\beta(n)|n-1\rangle \end{aligned} \tag{3.9}$$

Carrying out the quantum double theory on $SL_q(2)$ at q, a root of unity, and using the standard formula of Drinfeld we derive the following espression for the coloured R-matrix.

$$\begin{aligned} (R^{j_1(\lambda)j_2(\mu)})^{m_1' m_2'}_{m_1 m_2} &= q^{2(j_1+m_1'-(\gamma/2))(j_2+m_2'(-\mu/2))} \\ &\times\left\{\delta^{m_1'}_{m_1}\delta^{m_2'}_{m_2}+\sum_{n=0}^{k}\frac{(1-q^{-2})^n}{[n]!}q^{-\frac{1}{2}n(n-1)+n(j_1-j_2+m_1'-m_2'-(\gamma/2)+(\mu/2))}\right. \\ &\left.\times\prod_{1=0}^{n}\alpha_{j_1,m_1+l-1}(\lambda)\beta_{j_2,m_2-l+1}(\mu)[j_2+m_2-l+1]\delta^{m_1'}_{m_1+n}\delta^{m_2'}_{m_2-n}\right\} \\ &(\alpha p=2j+1,\, qP=1) \end{aligned} \tag{3.10}$$

Obviously if it satisfies the quantum double even more parameters appear in (3.10).

The details of the derivation of (3.10) can be found in [14.16]. Here we would like to emphasize that (3.10) is the consequence of 'mixture' between quantum algebra with q, a root of unity, and continuous parameter t as well as other colour parameters.

Having the general coloured solutions (3.10) we shall discuss the Markov trace properties. For simplicity we only deal with the case $j_1=j_2$. Since only $\lambda=\mu$ plays the role for coloured links [12], the R-matrix is simplified to

$$\begin{aligned} R^{m_1' m_2'}_{m_1 m_2} &= q^{\frac{1}{2}\{2(j+m_2')-\lambda\}\{2(j+m_1')-\lambda\}} \\ &\times\left\{\delta^{m_2'}_{m_1}\delta^{m_1'}_{m_2}+\sum_{n=1}^{2j}\frac{(1-q^{-2})^n}{[n]!}q^{-\frac{n(n-1)}{2}+n(m_2'-m_1')}\prod_{l=1}^{n}\alpha_{j,m_1+l-1}(\lambda)\right. \\ &\left.\times\beta_{j,m_2-l+1}(2)[j+m_2-l+1]\delta^{m_2'}_{m_1+n}\delta^{m_1'}_{m_2-n}\right\} \end{aligned} \tag{3.11}$$

where

$$\alpha_j, m-1(\lambda)\beta_{j,m}(\lambda)=[\lambda-j-m+1].$$

The Markov trace is sufficiently defined in terms of a diagonal matrix h such that [17]

$$\sum_N R^{mn}_{mn} h_{nn} \qquad \text{independent of } m \tag{3.12}$$

where

$$h_{mn} = h_m \delta_{mn} \text{ (no summation)} \tag{3.13}$$

The diagonal elements of (3.11) read

$$R^{m_1 m_2}_{m_1 m_2} = q^{\frac{1}{2}\{2(j+m_2)-\lambda\}\{2(j+m_1)-\lambda\}}$$
$$\times \left\{ \delta^{m_2}_{m_1} + \sum_{n=1}^{2j} \frac{(1-q^{-2})^n}{[n]!} q^{-(n(n-1)/2)+n(m_2-m_1)} \prod_{l=1}^{n} \alpha_{j,m_1+l-1}(\lambda)\beta_{j,m_2-l+1}(\lambda) \right.$$
$$\left. \times [j+m_2-l+1]\delta^{m_2}_{m_1+n} \right\} \tag{3.14}$$

Taking $\alpha_{j,m-1}(\lambda)\beta_{j,m}(\lambda) = [\lambda-j-m+1]$ into account and multiplying a common factor $q^{-\frac{1}{2}\lambda^2}$ the diagonal elements can be devided into two types

$$a_m \equiv R^{mm}_{mm} = q^{\frac{1}{2}\{2(j+m)-\lambda\}^2 - \frac{1}{2}\lambda^2} \tag{3.15}$$

and

$$b_{m,n} \equiv R^{mm+n}_{mm+n} = q^{\frac{1}{2}\{2(j+m+n)-\lambda\}\{2(j+m)-\lambda)\} - \frac{1}{2}\lambda^2}$$
$$\times \left\{ \frac{(1-q^{-2})^n}{[n]!} q^{\frac{1}{2}n(n+1)} \prod_{l=1}^{n} [\lambda-j-m-l+1][j+m+n-l+1] \right\}. \tag{3.16}$$

The equation (3.12) is equivalent ot he following $2j$ equations

$$R^{jj}_{jj}h_j = R^{j-1\,j-1}_{j-1\,j-1}h_{j-1} + R^{j-1\,j}_{j-1\,j}h_j = \ldots = \ldots$$
$$= R^{j-n\,j-n}_{j-n\,j-n}h_{j-n} + \sum_{i=1}^{n} R^{j-n\,j-n+i}_{j-n\,j-n+i}h_{j-n+i} = \ldots$$
$$= R^{-j\,-j}_{-j\,-j}h_{-j} + \sum_{i=1}^{2j} R^{-j\,i}_{-j\,i}h_{-j+i}. \tag{3.17}$$

Without loss of generality we take $h_j = 1$ then (3.17) reads

$$a_j = a_{j-1}h_{j-1} + b_{j-1,j}h_j = \ldots = \ldots$$
$$= a_{j-n}h_{j-n} + \sum_{k=1}^{n} b_{j-n,k}h_{j-n+k} = \ldots = \ldots$$
$$= a_{-j}h_{-j} + \sum_{k=1}^{2j} b_{-j,k}h_{-j+k}. \tag{3.18}$$

Since

$$q^p = 1 \qquad p = 4j+2 \qquad (\alpha = \tfrac{1}{2}) \tag{3.19}$$

we find

$$h_{j-1}=q^{4j}=q^{-2}. \tag{3.20}$$

In the following we shall prove that for the highest weight j and any other weight i, then h is given by (3.18). With the help of induction

$$h_{j-i}=q^{-2i} \tag{3.21}$$

we put the continuous parameter $t=q^{-\lambda}$ and $B^{(j,i)}=b_{j-i,k}h_{j-i+k}$ and substitute into

$$a_j=a_{j-i}h_{j-i}+\sum_{k=1}^{i} b_{j-i,k}h_{j-i+k} \tag{3.22}$$

to obtain

$$\sum_{k=1}^{i} B_k^{(j,i)}=q^{2(1+i)2-zi}t^{-2(1+i)}\sum_{k=1}^{i}\{q^{k(1-i)}\{{}^{i}_{k}\}_q(-1)^k\times\prod_{l=0}^{k-1}(1-t^2q^{zl-zi-2})\}. \tag{3.23}$$

By virtue of

$$\prod_{i=0}^{n-1}(1+q^{2i}z)=\sum_{i=0}^{n}\{{}^{n}_{i}\}_q q^{i(1-n)}z^i \tag{3.24}$$

where

$$\{{}^{n}_{i}\}=\frac{[n]!}{[i]!\,[n-i]!} \tag{3.25}$$

the term

$$\sum_{k=i}^{i}(-1)^k q^{k(1-i)}\{{}^{i}_{k}\}_q\sum_{l=0}^{k-1}(1-t^2q^{2l-zi-2}) \tag{3.26}$$

can be expanded in terms of t^2. The term with highest power is $q^{-2i(1+i)}t^{zi}$. The lowest term is constant

$$\sum_{k=1}^{i} q^{k(1-i)}\{{}^{i}_{k}\}_q(-1)^k=-1 \tag{3.27}$$

because of (3.25). The coefficients of t^{2m} $(1\leqslant m\leqslant i-1)$ vanish because

$$\sum_{k=m}^{i}(-1)^k q^{k(1-i)+mk}\{{}^{n}_{k}\}_q\{{}^{k}_{m}\}_q=0 \tag{3.28}$$

which can be verified by making the index translation

$$\sum_{k=0}^{I}(-1)^k q^{k-ki}\{{}^{i}_{k}\}_q=0\rightarrow\sum_{k=0}^{i-m}(-1)^k q^{k-ki+mk}\frac{1}{[i-k]!\,[k-m]!}=0. \tag{3.29}$$

Through calculation this can be recast to

$$\sum_{k=0}^{I}(-1)^k q^{k(1-i+m)}\frac{[i]!\,[k]!}{[k]!\,[i-k]!\,[k-m]!\,[m]!}=0. \tag{3.30}$$

Equation (3.30) means that the coefficients of t^2 $(1\leqslant m\leqslant i-1)$ vanish.

Substituting the above results into equation (3.22)

$$a_j = a_{j-i}h_{j-i} + \sum_{k=1}^{i} B_k^{(j,k)} \tag{3.31}$$

we confirm the validity of (3.21). Summing up all the calculations we conclude that the diagonal matrix h which is called the Markov trace [17] is ($\omega = q^2$)

$$h = \mathrm{diag}(1, \omega^{-1}, \omega^{-2}, \ldots, \omega^{-(N-1)}) \qquad \text{for } \omega^N = 1. \tag{3.32}$$

Obviously

$$\mathrm{tr}(h) = 0 \tag{3.33}$$

in particular for $j=1$ the parameter ω in (2.16) plays the role of q in (3.32) we thus have

$$h = \begin{bmatrix} 1 & & \\ & \omega^2 & \\ & & \omega \end{bmatrix} \tag{3.34}$$

For general soutions (3.11) we shall construct the Alexander link polynomials. To confirm the existence of such invariant tangles we should prove that the considered BGR is redundant in the sense of Murakami [12].

4. Reduncancy of equation (2.16)

The redundance of general solutions (3.11) are very complicated. In this paper we shall focus on the redundant property of (2.16) which is a special case of (3.11). For brevity we shall apply the theorum of Murakami literally as the following.

(1) Direct calculation gives that

$$\mathrm{tr}_2(\check{R} \cdot (I \otimes h)) = \alpha\beta I \tag{4.1}$$

$$\mathrm{tr}_2(\check{R}^{-1} \cdot (I \otimes h)) = \alpha^{-1}\beta I \tag{4.2}$$

where tr_2 means that the trace is taken on the second space. The $\check{R}$ is given by (2.16) and

$$\alpha = t^2 \qquad \beta = t^{-2}. \tag{44.3}$$

(2) Defining

$$r_i = \underbrace{I \otimes I \otimes \ldots \otimes I}_{i-1} \otimes R \otimes I \otimes \ldots \otimes I. \tag{4.4}$$

namely, $\{r_i\}$ is a representation of braid group. The Markov trace is defined by

$$\mathrm{tr}_{i,j} = \mathrm{tr}_i \cdot \mathrm{tr}_{i-1} \cdot \ldots \cdot \mathrm{tr}_j \qquad (i \geqslant j > 0) \tag{4.5}$$

for $\rho \in \mathrm{B}'$ generated by $\{r_i\}$. For instance

$$\mathrm{tr}_{i,j}(\rho) = \mathrm{tr}_i \cdot \mathrm{tr}_{i-l} \cdot \ldots \cdot \mathrm{tr}_j\{\rho(I \otimes \ldots \otimes h \otimes \ldots \otimes h \otimes I \otimes \ldots \otimes I)\} \tag{4.6}$$

(3) Let $Y=\{\check{R}, h, \alpha, \beta\}$ be an enchanced Yang-Baxter operator (EYB) and A_n be a subalgebra of End $(V\otimes V\otimes \ldots \otimes V)$ generated by the iamge $\rho(B_n)$ then the EYB-operator is termed redundant $\forall x\in A_n$

$$\mathrm{tr}_n\{X(I^{\otimes(n-1)}\otimes h)\}\in A_{n-1} \quad \text{for all } n>1. \tag{4.7}$$

The theorem states that if Y is redundant then

$$T(b)=\alpha^{-W(b)}\beta^{-n}\,\mathrm{tr}_{n,2}\{\rho(b)(I\otimes h\otimes \ldots \otimes h)\} \tag{4.8}$$

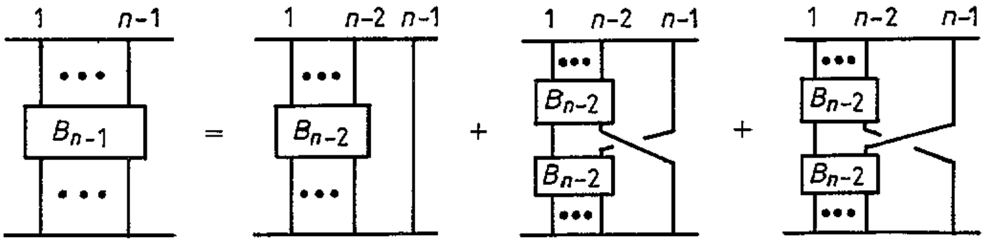
is an isotopy invariant of oriented links for $b\in B_n$ and $W(b)$ is determined by the writhe.

Now we shall point out that the EYB operator associated with (2.16) is redundant. The basic line of proof is the extension of the discussion in [6] but without using any properties of BWA

In [6] we have verified that if a braid block $A_n\in B_n$ satisfies

$$A_{n-1}=A_{n-2}+A_{n-2}r_{n-2}A_{n-2}+A_{n-2}r_{n-2}^{-1}A_{n-2} \tag{4.9}$$

then the corresponding Y must be redundant. BWA sufficiently satisfies this relation. Equation (4.9) can be graphically illustrated by



Now our task is to prove that r_{n-1} and r_{n-1}^{-1} appear only once in n-strings for the Y associated with (2.16). First we discuss the case with three strings then complete the proof by induction. Let us follow the standard procedure as presented in [12.6], that is essentially to find all independent bases.

The braid relations are

$$\begin{aligned} &b_i b_{i\pm 1} b_i = b_{i\pm 1} b_i b_{i\pm 1} \\ &b_i b_j = b_j b_i \qquad (|i-j|\geqslant 2) \end{aligned} \tag{4.10}$$

and the S given by (2.16) satisfies

$$\bar{S}^2=(\omega t^4-t^2+1)\bar{S}+(t^2+\omega t^6-\omega t^4)-\omega t^6\bar{S}^{-1}. \tag{4.11}$$

So the independent basis should be induced in the following listed ones

$$\begin{aligned} A_3=\{&I, r_1, r_2, r_1^{-1}, r_2^{-1}, r_1r_2, r_2r_1, r_1^{-1}r_2, r_2r_1^{-1}, r_1r_2^{-1}, r_2^{-2}r_1^{-1} \\ &r_2^{-1}r_1, r_1^{-1}r_2^{-1}, r_1r_2r_1, r_1^{-1}r_2^{-1}r_1^{-1}, r_1^{-1}r_2r_1, r_1r_2^{-1}r_1 \\ &r_1r_2r_1^{-1}, r_1r_2^{-1}r_1^{-1}, r_1^{-1}r_1^{-1}r_1, r_1^{-1}r_2r_1^{-1} \text{ and } r_2r_1^{-1}r_2, r_2^{-1}r_1r_2^{-1}\}. \end{aligned} \tag{4.12}$$

Observing (4.12) the r_2^{-1} and r_2 appear only once in the first 21 base so that (4.9) has already been satisfied because $r_1^{-1}, r_1\in B_1$. The difficulty is in the base is $r_2r_1^{-1}r_2$ (and its inverse) which formally does not obey (4.9). In the present case there is no algebraic relation like BWA to simplify computation. We have to directly check whether $r_2r_1^{-1}r_2$ can be expressed in terms of a linear combination of the other 21 bases.

A lengthy calculation gives

$$
\begin{aligned}
-r_2r_1^{-1}r_2 = {} & x_0I + x_1r_1 + x_2r_2 + x_3r_1^{-1} + x_4r_2^{-1} \\
& + x_5r_1r_2 + x_6r_2r_1 + x_7r_1^{-1}r_2 + x_8r_2r_1^{-1} + x_9r_1r_2^{-1} \\
& + x_{10}r_2^{-1}r_1 + y_1r_1^{-1}r_2^{-1} + y_2r_2^{-1}r_1^{-1} + y_3r_1r_2r_1 \\
& + y_4r_1^{-1}r_2^{-1}r_1^{-1} + y_5r_1r_2^{-1}r_1 + y_6r_1^{-1}r_2r_1 + y_7r_1r_2r_1^{-1} \\
& + y_8r_1r_2^{-1}r_1^{-1} + y_9r_1^{-1}r_2^{-1}r_1 + y_{10}r_1^{-1}r_2r_1^{-1}
\end{aligned}
\tag{4.13}
$$

where

$$
\begin{aligned}
& x_0 = -(t^2-1)^2(t^2\omega-1) - x_1 - x_3 \\
& x_1 = -\omega^2(t^2-\omega)(t^6-1)/\{t^2(t^4-t^2\omega+\omega^2)\} \\
& x_2 = -(t^2-1)(t^2\omega-1) - x_5 - x_7 \\
& x_3 = t^2(t^8+\omega^2t^6-t^4\omega+t^2\omega+\omega)/\{(t^2+1)(t^2\omega+1)\} \\
& x_4 = t^2(t^2-1)(t^2\omega+1) - y_1 - x_9 \\
& x_5 = x_6 = -\omega^2(t^8+t^4-t^2-\omega)/t^2(t^8-t^6\omega+t^2-\omega) \\
& x_7 = x_8 = (t^{12}+\omega^2t^{10}+t^8-\omega^2t^6+t^4-t^2-1)/\Delta \\
& x_9 = x_{10} = t^2-1-y_7-x_7 \\
& y_1 = y_2 = -\omega^2t^6(1-\omega^2t^8-t^6+t^4)/\Delta \\
& y_3 = -(y_7+x_5), \qquad y_5 = -(t^2+y_9+x_9) \\
& y_4 = -(y_1+y_9), \qquad y_6 = y_7 = -t^2(t^2\omega+\omega-1)/\Delta \\
& y_8 = y_9 = -t^6\{(\omega+2)t^4-\omega^2\}/\Delta \\
& y_{10} = t^2-1-y_7-x_7 = x_9 \\
& \Delta = -\omega^2(t^6+1)(t^2-\omega)
\end{aligned}
\tag{4.14}
$$

Based on (4.13) we conclude that any braid blocks associated with (2.16) for three strings satisfy (4.9), namely it is redundant. Now we show the statement works for n strings.

Lemma. Let A_{n-1} be associated with (2.16), and (4.9) is satisfied. Regarding A_{n-1} as a subalgebra of A_n then

$$A_n = A_{n-1} + A_{n-1}r_{n-1}A_{n-1} + A_{n-1}r_{n-1}^{-1}A_{n-1}. \tag{4.15}$$

The proof is on the basis of induction. For $n=2$ the set of basis of $A_2=\{I, r_1^{-1}, r_1, r_2, r_2^{-1}, r_1r_2, r_2r_1, r_1^{-1}r_2, r_2r_1^{-1}, r_1r_2^{-1}, r_2^{-1}, r_1, r_1^{-1}r_2^{-1}, r_2^{-1}r_1^{-1}\}$ the lemma is true.

For $n=3$ suppose (4.9) holds because $r_ir_j=r_jr_i(|i-j|\geqslant 2)$ we have

$$
\begin{aligned}
A_n = {} & A_{n-2} + A_{n-2}r_n^{\pm1} + A_{n-1}r_{n-1}^{\tau'}r_{n-2}r_{n-1}^{\tau}A_{n-1} \\
& + A_{n-1}r_{n-1}^{\tau'}, r_{n-2}^{-1}, r_{n-1}^{\tau}A_{n-1} \qquad (\tau, \tau'=\pm1, 0).
\end{aligned}
\tag{4.17}
$$

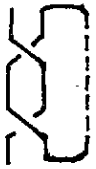
Since $r^{\tau'}r^{\pm1}r^{\tau}(\tau, \tau'=\pm1, 0)$ can be transformed into $r^{\tau}r^{\pm1}r^{\tau'}$, equation (4.17) is recast into (4.15). The lemma is proved.

Summing up the above discussions we conclude that the EYB operator Y associated with (2.16) is really redundant; even the S does not obey the BWA.

To graphically check our statement we list some invariant tangles calculated separately by

$$T(b)=\alpha^{-\omega(b)}\beta^{-n}\,\mathrm{tr}_{n,2}\{b(I\otimes h\otimes\ldots\otimes h)\} \tag{4.18}$$

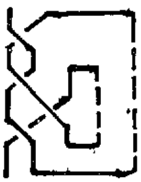
with



$$h=\begin{bmatrix}1 & & \\ & \omega^2 & \\ & & \omega\end{bmatrix},\qquad \alpha=t^2,\qquad \beta=t^{-2}. \tag{4.19}$$

$$\begin{aligned}&\mathrm{tr}_{n,2}\colon (t^2-1)(\omega t^2-1)\\ &T(r_1^2)=(t^2-1)(\omega t^2-1)\end{aligned} \tag{4.20}$$

$$\begin{aligned}&\mathrm{tr}_{n,2}\colon (t^2-1)(\omega t^2-1)\\ &T(r_1r_2r_1)=(t^2-1)(\omega t^2-1)\end{aligned} \tag{4.21}$$



$$\begin{aligned}&\mathrm{tr}_{n,2}\colon t^{-4}(t^2-1)(\omega t^2-1)\\ &T(r_2r_1^{-1}r_2)=(t^2-1)(\omega t^2-1)\end{aligned} \tag{4.22}$$

Observing (4.20), (4.21) and (4.22) we really obtain the same results for three invariant tangles.

Another example is given by

$$\begin{aligned}&\mathrm{tr}_{n,2}\colon -t^8(\omega+1)+t^6+t^4(2\omega+1)-t^2(\omega+1)+1\\ &T(r_1^2r_2r_1)=t^{-2}\{\omega^2t^8+t^6+t^4(2\omega+1)+\omega^2t^2+1\}.\end{aligned} \tag{4.23}$$

5. Concluding remarks

(1) The exotic family of solutions (3.10) is not a simple super extension of spin models. It originates from the 'continuous' extension of representations of quantum

algebra with q a root of unity. This extension preserves the quantum double of Drinfeld [3] and is emphasized by Jimbo [18].

(2) The diagonal matrix h in the Markov trace is explicitly given by direct calculations for non-coloured case.

(3) The 9×9 representation shown by (2.6) does not obey BWA. It can be Yang–Baxterized in the unique assignment of eigenvalues given by (2.16). We have verified that the EYB operator Y associated with (2.16) is redundant in the sense of Murakami for n-strings. Hence the related invariant tangles can be constructed explicitly.

(4) The Yang–Baxterization of (3.10) and their redundancy are attractive. The calculation is in progress. The corresponding state model is also interesting and is connected with an extension of the discussion of Kauffman and Saleur [9, 10].

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